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# ABEL'S THEOREM AND THE ADDITION FORMULAE FOR ELLIPTIC INTEGRALS

### BY HARRY HUNTINGTON BARNUM

THE addition formulae for the Legendrian elliptic integrals

$$1a)\qquad F(k, z) = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

$$1b)\qquad E(k, z) = \int_0^z \frac{(1-k^2z^2)\,dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

$$1c)\qquad \Pi(n, k, z) = \int_0^z \frac{dz}{(1+nz^2)\sqrt{(1-z^2)(1-k^2z^2)}},$$

are well known and have been derived in various ways. The object of the formulae is to give values for the expressions

$$F(k, z_1) + F(k, z_2),$$
$$E(k, z_1) + E(k, z_2),$$
$$\Pi(n, k, z_1) + \Pi(n, k, z_2),$$

each in terms of a third integral of the same type, whose upper limit is an algebraic function of $z_1$ and $z_2$, with the addition of an algebraic or a logarithmic term in the second and third cases.

In most of the elementary works on the subject the proofs of the formulae depend on the more or less artificial discovery of an algebraic solution of the differential equation

$$\frac{dz_1}{\sqrt{(1-z_1^2)(1-k^2z_1^2)}} + \frac{dz_2}{\sqrt{(1-z_2^2)(1-k^2z_2^2)}} = 0,$$

or upon somewhat complicated applications of Abel's Theorem. The object of this paper is to give an elementary derivation of the addition formulae suitable for presentation to a beginning class in the subject, by a method which may serve also as an introduction to Abel's famous theorem.

A complete comprehension of Abel's theorem as it is about to be stated* requires an acquaintance with the theory of functions of a complex variable, and in particular with Riemann surfaces. But the succeeding sections are developed independently of the theorem, its use being merely to indicate the methods to be used. Furthermore the lower limits of the integrals involved in the proofs are everywhere so arranged that only real expressions occur.

For the sake of brevity the addition formula for integrals of the first of the three types (1) is derived from that of the third type by putting $n = 0$. But the simple independent proof, quite similar to that of the third type can be readily carried through by the reader.

**1. Abel's Theorem.** Let $C$ and $C'$ be plane curves given by the equations

$$\begin{aligned} C &: F(xy) = 0, \\ C' &: \phi(xy) = 0. \end{aligned} \tag{2}$$

These curves have $n$ points of intersection $(x_1,y_1), \cdots (x_n,y_n)$, where $n$ is the product of the degrees of $C$ and $C'$. Let $R(x,y)$ be a rational function of $x$ and $y$ where $y$ is defined as a function of $x$ by the relation $F(x,y) = 0$.

Consider the sum

$$I = \sum_{i=1}^{n} \int_{x_0y_0}^{x_iy_i} R(x,y)\,dx, \tag{3}$$

the integrals $\int_{x_0y_0}^{x_iy_i} R(xy)\,dx$ being taken from a fixed point, $(x_0, y_0)$ in the Riemann surface of the function $F(x,y)$, to the $n$ points of intersection $(x_1, y_1), \cdots (x_n, y_n)$ of the curves $C$ and $C'$. If some of the coefficients $a_1, a_2, \cdots a_k$ of $\phi(x,y)$ are regarded as continuous variables, the points $(x_i,y_i)$ will vary continuously and hence $I$ will be a function, whose form is to be determined, of the variable coefficients $a_1, a_2, \cdots a_k$.

Abel's theorem may now be stated as follows:

The partial derivative of the sum $I$, with respect to any one of the coefficients of the variable curve $\phi = 0$, is a rational function of the coefficients and hence $I$ is equal to a rational function of the coefficients of

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* See Goursat, *Cours d'Analyse*, vol. 2, §360.

$\phi(x,y)=0$, plus a finite number of logarithms or arc tangents of such rational functions.

**2. A useful lemma.** Before proceeding to the derivation of the addition formulae a lemma* will be proved which will be of use later.

Let $f(x)=0$ be an equation of the $n$th degree with the distinct roots $x_1, x_2, \cdots x_n$. Also let $F(x)=0$ be any abitrary polynomial in $x$. The following identity can then be readily established,

$$\frac{xF(x)}{f(x)} = k + k_1x + k_2x^2 + \cdots + \sum_{i=1}^{n} \frac{x_iF(x_i)}{(x-x_i)f'(x_i)}. \tag{4}$$

The integral terms are the quotient of $xF(x)$ divided by $f(x)$, and the terms of the sum are the partial fractions to which the remainder over $f(x)$ gives rise. If we set $x=0$ we get

$$\sum_{i=1}^{n} \frac{F(x_i)}{f'(x_i)} = k \tag{5}$$

from (4). Hence we see that *if* $F(x)$ *is of degree* $(n-2)$ *or less, then* $k=0$, *and the symmetric functions* (5) *of the roots must be zero. If* $F(x)$ *is of degree* $(n-1)$ *or more*

$$\sum_{i=1}^{n} \frac{F(x_i)}{f'(x_i)}$$

*will be equal to the term independent of* $x$ *in the quotient* $xF(x)/f(x)$, as is indicated by (5).

**3. The Addition Formula for $E(k, z)$.** We proceed now to the establishment of the addition formula for elliptic integrals of the second type,

$$E(k, z) = \int_0^z \frac{(1-k^2z^2)\,dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{6}$$

By the substitution $z^2 = x$ (6) may be thrown into the more convenient form

$$E(k, x) = \int_0^x \frac{(1-k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}},$$

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* Petersen: *Funktionstheorie*, §39.

which is the Riemann normal form. We take now, corresponding to the curves $C$ and $C'$ of (2), the curves

$$(7a) \qquad C: \qquad y^2 = x(1-x)(1-k^2x),$$

$$(7b) \qquad C': \qquad y = ax + b.$$

The elimination of $y$ between these two equations will give us as the abscissae $x_1$, $x_2$, $x_3$ of the points of intersection the three roots of the equation

$$(8) \qquad \phi(x) = k^2x^3 - (1 + k^2 + a^2)\, x^2 + (1 - 2ab)x - b^2 = 0.$$

The function $R(x, y)$ in (3) is here

$$R(x, y) = \frac{1 - k^2x}{y},$$

where $y$ is defined by (7$a$). We have then corresponding to the sum (3), the function

$$(9) \qquad I(a, b) = \int_0^{x_1} \frac{(1 - k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2} \frac{(1 - k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} + \int_{\frac{1}{k^2}}^{x_3} \frac{(1 - k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}}.$$

For the purpose of keeping the discussion in the real field the lower limit of the third integral is taken as $1/k^2$ instead of 0. As however the integral between the limits 0 and $1/k^2$ is a constant, this is not an essential alteration of the sum which occurs in Abel's theorem.

To determine the form of $I(a, b)$ we take its partial derivatives with respect to $a$ and $b$

$$(10) \qquad \begin{aligned} \frac{\partial I}{\partial a} &= \frac{\partial I}{\partial x_1}\frac{\partial x_1}{\partial a} + \frac{\partial I}{\partial x_2}\frac{\partial x_2}{\partial a} + \frac{\partial I}{\partial x_3}\frac{\partial x_3}{\partial a} \\ &= \frac{1 - k^2x_1}{y_1}\frac{\partial x_1}{\partial a} + \frac{1 - k^2x_2}{y_2}\frac{\partial x_2}{\partial a} + \frac{1 - k^2x_3}{y_3}\frac{\partial x_3}{\partial a}. \end{aligned}$$

Since $x_1$ is a root of (8) we may substitute $x_1$ for $x$ in it and take the derivative with respect to $a$, noting that $x_1$ is a function of $a$. We find then

$$\phi'(x_1)\,\frac{\partial x_1}{\partial a} + \frac{\partial \phi(x_1)}{\partial a} = 0,$$

the accent denoting differentiation with respect to $x_1$. From equation (8) this becomes

$$\phi'(x_1)\,\frac{\partial x_1}{\partial a} - 2(ax_1+b)x_1 = 0,$$

or because of (7*b*)

$$\phi'(x_1)\,\frac{\partial x_1}{\partial a} - 2y_1x_1 = 0.$$

Substituting this value for $\frac{\partial x_1}{\partial a}$ and the similar ones for $\frac{\partial x_2}{\partial a}$ and $\frac{\partial x_3}{\partial a}$ in (10), we get

$$\frac{\partial I}{\partial a} = 2\sum_{i=1}^{3}\frac{x_i}{\phi'(x_i)} - 2k^2\sum_{i=1}^{3}\frac{x_i^2}{\phi'(x_i)}.$$

These sums are in the form of those in (5), where $f(x)$ is the function $\phi(x)$ in equation (8), and for the first sum $F(x) = x$, while for the second $F(x) = x^2$. Hence because of the lemma the first sum vanishes and the second is equal to the constant term $1/k^2$ of the quotient $x^3/\phi(x)$. We have therefore,

$$\frac{\partial I}{\partial a} = -2. \tag{11}$$

Similarly

$$\frac{\partial I}{\partial b} = \frac{\partial I}{\partial x_1}\frac{\partial x_1}{\partial b} + \frac{\partial I}{\partial x_2}\frac{\partial x_2}{\partial b} + \frac{\partial I}{\partial x_3}\frac{\partial x_3}{\partial b}$$

$$= 2\sum_{i=1}^{3}\frac{1}{\phi'(x_i)} - 2k^2\sum_{i=1}^{3}\frac{x_i}{\phi'(x_i)} = 0. \tag{12}$$

To find $I(a, b)$ we now integrate equations (11) and (12). Equation (11) gives

$$I(a, b) = -2a + f(b).$$

The derivative of this expression with respect to $b$ vanishes because of (12), and hence $f(b)$ must be a constant. Consequently equation (9) becomes

$$\int_0^{x_1}\frac{(1-k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2}\frac{(1-k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} + \int_{\frac{1}{k^2}}^{x_3}\frac{(1-k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} = -2a + C, \tag{13}$$

where $x_1$, $x_2$, $x_3$ are the abscissae of the intersections of the curve (7$a$) with the straight line (7$b$), i. e., the three roots of equation (8). The formula (13) is in fact an addition formula for $E(k, x)$, though not in the usual form.

In order to transform equation (13) let us keep the intersection point $(x_3, y_3)$ in figure 1 fixed, place $(x_1, y_1)$ at the origin, and denote the resulting

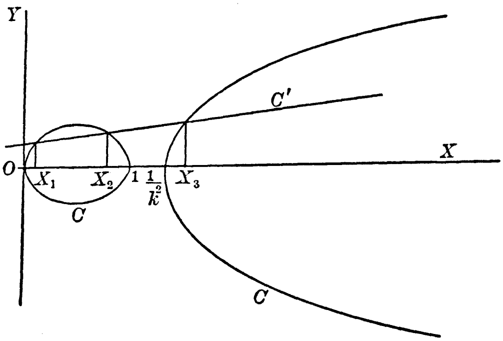

FIG. 1.

values of $x_2$, $y_2$ and $a$ by $x'$, $y'$ and $a'$ respectively. Then equation (13) becomes

$$\int_{\frac{1}{k^2}}^{x_3} \frac{(1 - k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} = -\int_0^{x'} \frac{(1 - k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} - 2a' + C,$$

which substituted in (13) gives

$$\int_0^{x_1} \frac{(1 - k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2} \frac{(1 - k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}}$$
$$= \int_0^{x'} \frac{(1 - k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} + 2(a' - a),$$

the usual form for the addition theorem.

It remains only to express $x'$ and $a' - a$ in terms of $x_1$ and $x_2$. Since $x_1$, $x_2$, $x_3$ are the roots of equation (8), we have

$$(14)\quad x_1x_2x_3 = \frac{b^2}{k^2},\quad x_2x_3 + x_3x_1 + x_1x_2 = \frac{1 - 2ab}{k^2},\quad x_1 + x_2 + x_3 = \frac{1 + k^2 + a^2}{k^2},$$

from which we find, when $x_1 = 0$,

$$b = 0, \quad x' = \frac{1}{k^2 x_3}, \quad a' = \sqrt{\frac{(1-x')(1-k^2x')}{x'}}. \tag{15}$$

From equations (7*b*) and (14)

$$\begin{aligned} a &= \frac{y_3}{x_3} - \frac{b}{x_3} = \sqrt{\frac{(1-x_3)(1-k^2x_3)}{x_3}} - k\sqrt{\frac{x_1x_2}{x_3}} \\ &= \sqrt{\frac{(1-x')(1-k^2x')}{x'}} - k^2\sqrt{x_1x_2x'} \\ &= a' - k^2\sqrt{x_1\, x_2\, x'}. \end{aligned} \tag{16}$$

From equations (14) and (15)

$$x' = \frac{1}{k^2x_3} = \frac{x_1x_2}{b^2}.$$

Since $(x_1, y_1)$ and $(x_2, y_2)$ are on the straight line (7*b*), the value of $b$ can be found from the equations

$$y_1 = ax_1 + b, \quad y_2 = ax_2 + b.$$

We have then

$$b = \frac{x_1y_2 - x_2y_1}{x_1 - x_2} = x_1\, x_2\, \frac{1 - k^2x_1x_2}{x_1y_2 + x_2y_1},$$

where the second form is found by rationalizing the numerator of the first by the help of equation (7*a*), which $(x_1, y_1)$ and $(x_2, y_2)$ also satisfy. The final expression for $x'$ is therefore

$$x' = \frac{1}{x_1x_2}\left(\frac{x_1y_2 + x_2y_1}{1 - k^2x_1x_2}\right)^2. \tag{17}$$

*The addition formula for Elliptic integrals of the second type in the Riemann normal form is*

$$\begin{aligned} &\int_0^{x_1} \frac{(1-k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2} \frac{(1-k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} \\ &\qquad = \int_0^{x'} \frac{(1-k^2x)\,dx}{\sqrt{x(1-x)(1-k^2x)}} + 2k^2\sqrt{x_1x_2x'}, \end{aligned} \tag{18}$$

*where $x'$ is defined in terms of $x_1$ and $x_2$ by equation* (17). The addition formula for the Legendre normal form can be found by transforming the variables in (17) and (18) by the substitution $x = z^2$.

It follows that *the addition formula for $E(k, z)$ is*

$$(19)\qquad \int_0^{z_1} \frac{(1-k^2z^2)\,dz}{\sqrt{(1-z^2)(1-k^2z^2)}} + \int_0^{z_2} \frac{(1-k^2z^2)\,dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \int_0^{z} \frac{(1-k^2z^2)\,dz}{\sqrt{(1-z^2)(1-k^2z^2)}} + k^2z_1z_2z',$$

*where $z'$ is expressible in the form*

$$(20)\qquad z' = \frac{z_1\sqrt{(1-z_2^2)(1-k^2z_2^2)} + z_2\sqrt{(1-z_1^2)(1-k^2z_1^2)}}{1-k^2z_1^2z_2^2}.$$

The addition formula for the usual trigonometric form of $E(k, z)$ can be readily found by making the substitution $z = \sin\phi$ on all the variables in (19) and (20).

**4. The addition formula for** $\Pi(n, k, z)$. We proceed now to the treatment of elliptic integrals of the third type, with the help again of the Riemann normal form. We take as the curves $C$ and $C'$ in (2) the same curves (7), the function $R(x, y)$ being in this case, however,

$$R(x, y) = \frac{1}{(1+nx)y}.$$

The sum analagous to that occurring in Abel's theorem is here

$$(21)\quad K(ab) = \int_0^{x_1} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} + \int_{\frac{1}{k^2}}^{x_3} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}}.$$

The partial derivatives of $K$ with respect to $a$ and $b$ are

$$(22)\qquad \begin{aligned}\frac{\partial K}{\partial a} &= \frac{\partial K}{\partial x_1}\frac{\partial x_1}{\partial a} + \frac{\partial K}{\partial x_2}\frac{\partial x_2}{\partial a} + \frac{\partial K}{\partial x_3}\frac{\partial x_3}{\partial a}\\ &= 2\sum_{i=1}^{3} \frac{x_i}{(1+nx_i)\phi'(x_i)}.\end{aligned}$$

In order to apply the lemma in (5) we take

$$F(x) = x,$$
$$f(x) = k^2(1 + nx)(x - x_1)(x - x_2)(x - x_3),$$

with the roots $-1/n$, $x_1$, $x_2$, $x_3$. We have then

$$\sum_{i=1}^{n} \frac{F(x_i)}{f'(x_i)} = \frac{\frac{1}{n}}{k^2 n\left(\frac{1}{n} + x_1\right)\left(\frac{1}{n} + x_2\right)\left(\frac{1}{n} + x_3\right)} + \sum_{i=1}^{3} \frac{x_i}{(1 + nx_i)\phi'(x_i)} .$$

According to the lemma this sum must be zero, which gives us

$$\sum_{i=1}^{3} \frac{x_i}{(1 + nx_i)\phi'(x_i)} = \frac{-n}{k^2(1 + nx_1)(1 + nx_2)(1 + nx_3)} .$$

The substitution of this value in (22) gives

$$\frac{\partial K}{\partial a} = \frac{-2n}{k^2(1 + nx_1)(1 + nx_2)(1 + nx_3)} .$$

This may be written in the form

$$\frac{\partial K}{\partial a} = \frac{-2n}{k^2[1 + n(x_1 + x_2 + x_3) + n^2(x_2x_3 + x_3x_1 + x_1x_2) + n^3x_1x_2x_3]} .$$

The application of the relations given in (14) reduces this to

$$(23) \qquad \frac{\partial K}{\partial a} = \frac{-2}{a^2 - 2nba + 1 + k^2 + n + nb^2 + \frac{k^2}{n}} .$$

In a similar manner, by using $F(x) = 1$ in the lemma, the value of $\frac{\partial K}{\partial b}$ is found to be

$$(24) \qquad \frac{\partial K}{\partial b} = \frac{2n}{n^2b^2 - 2anb + n + 1 + k^2 + a^2 + \frac{k^2}{n}} .$$

The integration of equation (23) can be effected by the use of a table of integrals. If we confine ourselves to real variables, there are two cases according as $p = (1+n)(1+k^2/n)$ is greater than or less than zero. In the former case, we have

$$(25) \qquad p > 0, \quad K = \frac{2}{\sqrt{p}} \operatorname{arc\,tan} \frac{nb-a}{\sqrt{p}} + f_1(b).$$

For the latter case,

$$(26) \qquad p < 0, \quad K = \frac{1}{\sqrt{-p}} \log \frac{nb-a-\sqrt{-p}}{nb-a+\sqrt{-p}} + f_2(b).$$

By means of the formula

$$\operatorname{arc\,tan} x = \frac{1}{2i} \log \frac{i+x}{i-x}, \qquad i = \sqrt{-1}$$

the two cases could be treated simultaneously. As it is our object however to keep the discussion in the real field the two cases will be treated separately.

Differentiation of equations (25) and (26) with respect to $b$, gives us

$$\frac{\partial K}{\partial b} = \frac{2n}{n^2b^2 - 2anb + n + 1 + k^2 + a^2 + \dfrac{k^2}{n}} + f'(b),$$

and comparison with equation (24) shows us that $f'(b) = 0$, and hence $f(b) = C$ in each case.

From this point the work will be carried through only for the function (25), since the method is the same for each. Equation (21) then becomes

$$(27) \qquad \begin{aligned} &\int_0^{x_1} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} \\ &\qquad + \int_{\frac{1}{k^2}}^{x_3} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} = \frac{2}{\sqrt{p}} \operatorname{arc\,tan} \frac{nb-a}{\sqrt{p}} + C. \end{aligned}$$

Placing the point $(x_1, y_1)$ at the origin, and using the notation of §3, we find again a value for the third integral in this expression:

$$\begin{aligned} \int_{\frac{1}{k^2}}^{x_3} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} &= -\int_0^{x'} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} \\ &\qquad + \frac{2}{\sqrt{p}} \operatorname{arc\,tan} \frac{-a'}{\sqrt{p}} + C. \end{aligned}$$

Therefore (27) becomes

$$\int_0^{x_1} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}}$$
$$= \int_0^{x'} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} + \frac{2}{\sqrt{p}}\left[\text{arc tan}\,\frac{nb-a}{\sqrt{p}} + \text{arc tan}\,\frac{a'}{\sqrt{p}}\right].$$

The trigonometric formula

$$\text{arc tan}\,\alpha + \text{arc tan}\,\beta = \text{arc tan}\,\frac{\alpha+\beta}{1-\alpha\beta}$$

reduces the term outside the integral signs to the following form:

$$\text{arc tan}\,\frac{nb-a}{\sqrt{p}} + \text{arc tan}\,\frac{a'}{\sqrt{p}} = \text{arc tan}\,\frac{\sqrt{p}(nb-a+a')}{p-a'(nb-a)}.$$

The substitution of the values of $a$, $a'$, and $b$ from (14), (15), and (16) and the division of both numerator and denominator by the common factor $(n+k^2x')$ gives as *the final form of the addition formula of Elliptic integrals of the third type in the Riemann normal form, when* $p > 0$:

$$\int_0^{x_1} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}} + \int_0^{x_2} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}}$$
$$= \int_0^{x'} \frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}}$$
$$+ \frac{2}{\sqrt{p}}\,\text{arc tan}\,\frac{\sqrt{p}\,n\sqrt{x_1x_2x'}}{1+nx'-n\sqrt{x_1x_2(1-x')(1-k^2x')}},$$

*where* $x'$ *is expressible in terms of* $x_1$ *and* $x_2$ *by equation* (17).

The substitution $x = z^2$ as before gives us the corresponding *addition formula for the Legendre form*:

$$\int_0^{z_1} \frac{dz}{(1+nz^2)\sqrt{(1-z^2)(1-k^2z^2)}} + \int_0^{z_2} \frac{dz}{(1+nz^2)\sqrt{(1-z^2)(1-k^2z^2)}}$$
$$= \int_0^{z'} \frac{dz}{(1+nz^2)\sqrt{(1-z^2)(1-k^2z^2)}}$$
$$+ \frac{1}{\sqrt{p}}\,\text{arc tan}\,\frac{\sqrt{p}\,nz_1z_2z'}{1+nz'^2-nz_1z_2\sqrt{(1-z'^2)(1-k^2z'^2)}},$$

where the expression for $z'$ in terms of $z_1$ and $z_2$ is given in (20).

A similar treatment of the case where $p < 0$ brings us to the forms:

$$\int_0^{x_1}\frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}}+\int_0^{x_2}\frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}}$$

$$=\int_0^{x'}\frac{dx}{(1+nx)\sqrt{x(1-x)(1-k^2x)}}$$

$$+\frac{1}{\sqrt{-p}}\log\frac{n\sqrt{x_1x_2(1-x')(1-k^2x')}-(1+nx')-\sqrt{-p}\,n\sqrt{x_1x_2x'}}{n\sqrt{x_1x_2(1-x')(1-k^2x')}-(1+nx')+\sqrt{-p}\,n\sqrt{x_1x_2x'}},$$

and

$$\int_0^{z_1}\frac{dz}{(1+nz^2)\sqrt{(1-z^2)(1-k^2z^2)}}+\int_0^{z_2}\frac{dz}{(1+nz^2)\sqrt{(1-z^2)(1-k^2z^2)}}$$

$$=\int_0^{z'}\frac{dz}{(1+nz^2)\sqrt{(1-z^2)(1-k^2z^2)}}$$

$$+\frac{1}{2\sqrt{p}}\log\frac{nz_1z_2\sqrt{(1-z'^2)(1-k^2z'^2)}-(1+nz'^2)-\sqrt{-p}\,nz_1z_2z'}{nz_1z_2\sqrt{(1-z'^2)(1-k^2z'^2)}-(1+nz'^2)+\sqrt{-p}\,nz_1z_2z'}.$$

As stated at the beginning of the paper, the addition formula for Elliptic integrals of the first type may be found directly from those of the third type by putting $n = 0$. From (23) and (24) we see that if $n = 0$, then $K(ab)$ is a constant, and hence *the addition formula for Elliptic integrals of the first type reduces to the simple forms*:

$$\int_0^{x_1}\frac{dx}{\sqrt{x(1-x)(1-k^2x)}}+\int_0^{x_2}\frac{dx}{\sqrt{x(1-x)(1-k^2x)}}=\int_0^{x'}\frac{dx}{\sqrt{x(1-x)(1-k^2x)}},$$

and

$$\int_0^{z_1}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}+\int_0^{z_2}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}=\int_0^{z'}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

where, as before, $x'$ and $z'$ are related to $x_1$, $x_2$ and $z_1$, $z_2$ respectively by equations (17) and (20).

PRINCETON, UNIVERSITY,
PRINCETON, N. J.